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THE EFFECT OF THE BATDORF PARAMETER ON THE POST-CRITICAL BEHAVIOUR OF AN AXIALLY COMPRESSED IMPERFECT CYLINDRICAL SHELL†

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The loss of stability and post-critical behaviour of a geometrically imperfect elastic cylindrical shell subjected to axial compression at moveable hinged endfaces are asymptotically analysed in the limit as $Z \to \infty$ (where Z is the Batdorf parameter). The asymptotic behaviour of the eigenvalues and associated vectorial eigenfunctions, linearized about a torqueless solution of the boundary-value problem are constructed when $Z \to \infty$. The Lyapunov–Schmidt method is applied in the neighbourhood of each eigenvalue for which the asymptotic behaviour has been determined. For $Z \to \infty$ equilibrium eigenshapes that are odd with respect to the axial coordinate are shown to be unstable (the Koiter parameter $b < 0$), and the even ones ($b > 0$) are shown to be stable. It is shown that by an appropriate choice of initial imperfection the upper critical load for shell loss of stability (the limiting point) can be made to correspond to any of the close to ($Z \to \infty$) critical loads for loss of stability of an ideal shell.

1. STATEMENT OF THE PROBLEM

Based on the non-linear Mushtari–Donnell–Vlasov medium deflection theory of gently-curving shells, the equilibrium of an elastic circular-cylindrical shell under the action of a uniformly distributed axial compressive load, taking into account small imperfection in its shape, can be described by the system of partial differential equations

$$\begin{cases} \varepsilon^2\Delta^2 W + PW,_{xx} - F,_{xx} + \xi P\zeta,_{xx} - [W, F] - \xi[\zeta, F] = 0 \\ \varepsilon^2\Delta^2 F + W,_{xx} + [W, W]/2 + \xi[\zeta, W] = 0 \end{cases} \tag{1.1}$$

$$\Delta() = (),_{xx} + (),_{yy}; \ [W, F] = W,_{xx} F,_{yy} - 2W,_{xy} F,_{xy} + W,_{yy} F,_{xx}$$

$$\varepsilon^2 = Rh/(\gamma L^2) = (\sqrt{12}Z)^{-1}, \quad \gamma = \sqrt{12(1-\nu^2)}, \quad l = 2\pi R/L$$

in the domain $G = \{(x, y): |x| < 1/2; |y| < l/2\}$.

We shall consider system (1.1) together with boundary conditions

$$W = 0, \quad W,_{xx} = 0, \quad F = 0, \quad F,_x = 0 \ \text{when} \ |x| = 1/2 \tag{1.2}$$

The following notation is used: xL, yL are respectively the axial and circular coordinates, WL^2/R is the additional deflection, $Eh^2L^2R^{-1}\gamma^{-1}(F - Py^2/2)$ is the stress function, $\xi\zeta L^2R$ is the initial imperfection function ($|\zeta| \ll 1$), $PP_*/2$ is the axial load parameter, $P_* = 2Eh/\gamma R$ is the classical critical load, ε^2 is the relative thinness parameter, Z is the Batdorf parameter, L, r and h are the length, radius of curvature and thickness of the shell, E is Young's modulus and ν is Poisson's ratio.

Assuming that the function ζ is sufficiently smooth in G, the boundary-value problem (1.1), (1.2) will be treated as a non-linear operator equation

$$V(\mathbf{u}, P, \xi\zeta, \varepsilon^2) = 0, \quad V: H_4 \to Y_2 \tag{1.3}$$

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Here

$$\mathbf{u} = (W, F), \quad V(\mathbf{u}, P, \xi\zeta, \varepsilon^2) = M(P, \varepsilon^2)\mathbf{u} + N(\mathbf{u}, P, \xi\zeta)$$

$$M(P,\varepsilon^2)() = \begin{Vmatrix} \varepsilon^2\Delta^2() + P()_{,xx} & -()_{,xx} \\ -()_{,xx} & -\varepsilon^2\Delta^2() \end{Vmatrix}$$

$$N(\mathbf{u},P,\xi\zeta) = \begin{Vmatrix} \xi P\zeta_{,xx} - [W,F] - \xi[\zeta,F] \\ -[W,W]/2 - \xi[\zeta,W] \end{Vmatrix} \tag{1.4}$$

Y_2 is the linear space of two-dimensional vector functions $\mathbf{f} = (f_1, f_2)$, $\mathbf{g} = (g_1, g_2), \ldots$ with finite norm generated by the scalar product

$$\langle \mathbf{f}, \mathbf{g} \rangle_2 = \int_G (f_1 g_1 + f_2 g_2)\, dxdy \tag{1.5}$$

and the space H_4 is the closure of the set of vector functions $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (\upsilon_1\ \upsilon_2), \ldots$ that are infinitely differentiable in G, satisfy boundary conditions (1.2), and are finite in the norm, generated by the scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle_4 = \varepsilon^3 \langle \Delta\mathbf{u}, \Delta\mathbf{v} \rangle_2 \tag{1.6}$$

The zero-moment equilibrium shape $(W_*, F_*) = (0, Py^2/2)$ of the shell in (1.3) when $\xi = 0$ corresponds to the trivial solution. This solution can lose stability at the bifurcation point of $P = P_0$ of the operator $V(\xi = 0)$.

We note that when $P = P_0$ the operator $M(P, \varepsilon^2)$ has a zero eigenvalue (EV). This value of the parameter P is called critical. The presence of small imperfections means that the fundamental zero-moment equilibrium shape loses its stability not at the bifurcation point, but at a limiting point. It is important to investigate the effect of the Batdorf parameter (large for thin-walled shells) on the loss of stability for the basic equilibrium shape of the shell, and to determine the bifurcation values of the axial loading parameter and the upper critical load for loss of stability (the limiting point) due to the presence of small imperfections.

2. ASYMPTOTIC FORMS OF THE CRITICAL VALUES OF THE PARAMETER P WHEN $\varepsilon \to 0$

The critical values of the parameter P are EVs of the boundary-value problem

$$\begin{gathered} \varepsilon^2\Delta^2 W + PW_{,xx} - F_{,xx} = 0, \quad \varepsilon^2\Delta^2 F + W_{,xx} = 0 \\ W(\pm{}^1/_2) = 0, \quad W_{,xx}(\pm{}^1/_2) = 0, \quad F(\pm{}^1/_2) = 0, \quad F_{,x}(\pm{}^1/_2) = 0 \end{gathered} \tag{2.1}$$

Here the vector eigenfunctions (VEFs) $\mathbf{S} = (W, F)$ should be periodic with period l in the variable y. We shall construct the asymptotic forms of the EV $P \in [0, 2)$ and its associated VEF as $\varepsilon \to 0$. We will first consider the case when VEF does not depend on y. Putting $x = \varepsilon t$, $W = u_0$ and $F = \upsilon_0$, in this case we have from (2.1)

$$\begin{gathered} u_0^{\mathrm{IV}} + Pu_0'' - \upsilon_0'' = 0, \quad \upsilon_0^{\mathrm{IV}} + u_0'' = 0, \quad ()' = d()/dt \\ u_0(\pm\Lambda) = 0, \quad u_0''(\pm\Lambda) = 0, \quad \upsilon_0(\pm\Lambda) = 0, \upsilon_0'(\pm\Lambda) = 0, \quad \Lambda = 1/(2\varepsilon) \end{gathered} \tag{2.2}$$

We denote by $P_0^{(0)}$ ($P_0^{(1)}$) the EVs which correspond to the even (odd) VEFs $\mathbf{S}_0^{(0)}$ ($\mathbf{S}_0^{(1)}$) of problem (2.2). When $P \in [0, 2)$ we shall seek the solution of problem (2.2) in the form

$$\mathbf{S}_0^{(0)}(t) = 2\,\mathrm{Re}(\mathbf{A}\,\mathrm{ch}\,\eta t/\mathrm{ch}\,\eta\Lambda) + \mathbf{a}, \quad \mathbf{A} = A_1(1, -2\eta^{-2}), \quad \mathbf{a} = (a_1, a_2)$$

$$\mathbf{S}_0^{(1)}(t) = 2\,\mathrm{Re}(\mathbf{B}\,\mathrm{sh}\,\eta t / \mathrm{sh}\,\eta\Lambda) + \mathbf{c}t, \quad \mathbf{B} = B_1(1, -2\eta^{-2}), \quad \mathbf{c} = (c_1, c_2) \tag{2.3}$$

$$\eta = (s_1 + is_2)/\sqrt{2}, \quad s_1 = \sqrt{1-\rho}, \quad s_2 = \sqrt{1+\rho}, \quad \rho = P/2$$

where the complex constants A_1 and B and the real constants a_j, c_i ($i = 1, 2$) are for the time being unknown. Substituting (2.3) in turn into the boundary conditions of problem (2.2), we obtain an inhomogeneous system of linear algebraic equations whose non-trivial solvability leads to the transcendental equation

$$\begin{gathered}(1-P)s_2 + [2(1+P)s_1\sin(\sqrt{2}s_2\Lambda) - (1-P)s_2\exp(-\sqrt{2}s_2\Lambda)]\exp(-\sqrt{2}s_1\Lambda) = 0 \\ 1 - P + 2\sqrt{2}Ps_1\varepsilon + [(P-1)s_2\exp(-\sqrt{2}s_2\Lambda) + P - 1 - 2\sqrt{2}Ps_1\varepsilon + \\ +4\sqrt{2}Ps_1\varepsilon\sin(\sqrt{2}s_2\Lambda)]\exp(-\sqrt{2}s_1\Lambda) = 0\end{gathered} \tag{2.4}$$

It can be shown that for sufficiently small $\varepsilon > 0$ each of Eqs (2.4) has a single root $\rho \in [0, 1)$. Constructing the asymptotic forms of these roots as $\varepsilon \to 0$, we find that for the corresponding EVs of problem (2.2)

$$\begin{gathered}P_0^{(0)} = 1 + \frac{4}{\sqrt{3}}\sin\frac{\sqrt{3}}{2\varepsilon}\exp\left(-\frac{1}{2\varepsilon}\right) + o\left(\exp\left(-\frac{1}{2\varepsilon}\right)\right) \\ P_0^{(1)} = 1 + 2\varepsilon + 2\varepsilon^2 + O(\varepsilon^3)\end{gathered} \tag{2.5}$$

We will now consider the case when the VEFs of problem (2.1) depend on y. We shall seek the solution of problem (2.1) in the form

$$\begin{gathered}\mathbf{S}(x, y) = \mathbf{S}_n\left(\frac{x}{\varepsilon}\right)\sin\left(d_n y + \frac{\pi n}{2}\right) \\ \mathbf{S}_n\left(\frac{x}{\varepsilon}\right) = \left(u_n\left(\frac{x}{\varepsilon}\right), \upsilon_n\left(\frac{x}{\varepsilon}\right)\right), \quad d_n = \frac{2\pi n}{l}\end{gathered} \tag{2.6}$$

Using separation of variables after the substitution $x = \varepsilon t$, (2.1) gives

$$\begin{gathered}\Phi(\theta_n)u_n + Pu_n'' - \upsilon_n'' = 0, \quad \Phi(\theta_n)\upsilon_n + u_n'' = 0 \\ u_n(\pm\Lambda) = 0, \quad u_n''(\pm\Lambda) = 0, \quad \upsilon_n(\pm\Lambda) = 0, \quad \upsilon_n'(\pm\Lambda) = 0, \quad n = 1, 2, \ldots \\ \Phi(\theta_n)() = ()^{\mathrm{IV}} - 2\theta_n^2()'' + \theta_n^4(), \quad \theta_n = d_n\varepsilon\end{gathered} \tag{2.7}$$

Searching for a solution of problem (2.7) of the form

$$u_n(t) = -(r^2 - \theta_n^2)^2\exp(rt), \quad \upsilon_n(t) = r^2\exp(rt)$$

we arrive at the characteristic equation

$$(r^2 - \theta_n^2)^2 / r^2 + r^2 / (r^2 - \theta_n^2)^2 = -2\rho \tag{2.8}$$

which has four pairs of complex roots $\pm\eta_j$, $\pm\bar{\eta}_j$ ($j = 1, 2$) where

$$\begin{gathered}\eta_1 = (s_1 + t_1 + i(s_2 + t_2))/(2\sqrt{2}), \quad \eta_2 = (s_1 - t_1 + i(s_2 - t_2))/(2\sqrt{2}) \\ t_1 = \sqrt{R + 4\theta_n^2 - \rho}, \quad t_2 = \sqrt{R - 4\theta_n^2 + \rho}, \quad R = \sqrt{16\theta_n^4 - 8\rho\theta_n^2 + 1}\end{gathered} \tag{2.9}$$

We shall seek the even solutions $\mathbf{S}_n^{(0)}(t)$ and odd solutions $\mathbf{S}_n^{(1)}(t)$ of problem (2.7) in the respective forms

$$\mathbf{S}_n^{(0)}(t) = 2\,\mathrm{Re}\left(\sum_{j=1}^{2} \mathbf{C}_j \frac{\operatorname{ch}\eta_j t}{\operatorname{ch}\eta_j\Lambda}\right), \quad \mathbf{C}_j = C_{j,1}(-1,\overline{C})$$

$$\mathbf{S}_n^{(1)}(t) = 2\,\mathrm{Re}\left(\sum_{j=1}^{2} \mathbf{D}_j \frac{\operatorname{sh}\eta_j t}{\operatorname{sh}\eta_j\Lambda}\right), \quad \mathbf{D}_j = D_{j,1}(-1,\overline{C}) \tag{2.10}$$

$$C = -\rho + is_1s_2$$

We arrive at equations similar to (2.4)

$$\mathrm{Im}[C(\eta_1^2-\eta_2^2)(\overline{\eta}_1 \operatorname{th}\overline{\eta}_1\Lambda - \overline{\eta}_2 \operatorname{th}\overline{\eta}_2\Lambda)] = 0$$

$$\mathrm{Im}[C(\eta_1^2-\eta_2^2)(\overline{\eta}_1 \operatorname{cth}\overline{\eta}_1\Lambda - \overline{\eta}_2 \operatorname{cth}\overline{\eta}_2\Lambda)] = 0 \tag{2.11}$$

It can be shown that for sufficiently small $\varepsilon > 0$ each of Eqs (2.11) has precisely one root $\rho \in [0, 1)$. Constructing the asymptotic forms of these roots as $\varepsilon \to 0$, we obtain the behaviour of the corresponding EVs of problem (2.7)

$$P_n^{(0)} = 1 + d_n^2\varepsilon^2 + O(\varepsilon^3)$$

$$P_n^{(1)} = 1 + 2\varepsilon + (2 + d_n^2)\varepsilon^2 + O(\varepsilon^3), \quad n = 1,2,\ldots \tag{2.12}$$

The asymptotic formula (2.12) can be proved by methods similar to those described previously [1]. As a result we arrive at the following theorem.

Theorem 1. Let $\varepsilon > 0$ be a sufficiently small number. Then for every positive integer $n < n_*$, where $n^* = \sqrt{(3)}l/(4\sqrt{(2)}\pi\varepsilon)$ (the values in square brackets represent the integer part of a number), the boundary-value problem (2.1) has exactly two EVs $P_n^{(0)}, P_n^{(1)} \in [0, 2)$. Here the following assertions hold:

1. for a given natural k one can find an $\varepsilon_0 > 0$ such that when $0 < \varepsilon < \varepsilon_0$ and $0 \leqslant n \leqslant k$, the EVs $P_n^{(j)}$ $(j = 0, 1)$ have the asymptotic representations (2.5) and (2.12)
2. $P_n^{(0)} < P_{n+1}^{(0)}$; $P_n^{(1)} < P_{n+1}^{(1)}$; $P_n^{(0)} < P_n^{(1)}$ $(n = 1, 2, \ldots, n_* - 1)$.

We will now give formulae for the VEFs, corresponding to the EVs from (2.5) and (2.12)

$$u_0^{(0)}(t) = 2\,\mathrm{Re}\left(A_1 \frac{\operatorname{ch}\eta t}{\operatorname{ch}\eta\Lambda}\right) - 1, \quad \upsilon_0^{(0)}(t) = 2\,\mathrm{Re}\left(A_1\eta^{-2} \frac{\operatorname{ch}\eta t}{\operatorname{ch}\eta\Lambda}\right) - \rho$$

$$\rho = P_0^{(0)}/2, \quad A_1 = (1-2i\rho)/(2s_1s_2)$$

$$u_0^{(1)}(t) = 2\,\mathrm{Re}\left(B_1 \frac{\operatorname{sh}\eta t}{\operatorname{sh}\eta\Lambda}\right) + t, \quad \upsilon_0^{(1)}(t) = 2\,\mathrm{Re}\left(B_1\eta^{-2} \frac{\operatorname{sh}\eta t}{\operatorname{sh}\eta\Lambda}\right) - 2\rho t$$

$$\rho = P_0^{(1)}/2; \quad B_1 = \Lambda(\tfrac{1}{2} - i\rho)/(2s_1s_2)$$

$$\mathbf{S}_n^{(j)}(t) = J(Q\overline{Y}_j(t) - Y_j(t),\ -QC\overline{Y}_j(t) + \overline{C}Y_j(t)), \quad j = 0,1$$

$$Q = Q_1 + iQ_2, \quad Q_1 = R^{-1}(2\rho^2 - 4\theta_n^2\rho - 1), \quad Q_2 = R^{-1}s_1s_2(4\theta_n^2 - 2\rho)$$

$$J = 1 + iQ_2/(1-Q_1), \quad \rho = P_n^{(j)}/2, \quad j = 0,1$$

$$Y_0(t) = \frac{\operatorname{ch}\eta_1 t}{\operatorname{ch}\eta_1\Lambda} - \frac{\operatorname{ch}\eta_2 t}{\operatorname{ch}\eta_2\Lambda}, \quad Y_1(t) = \frac{\operatorname{sh}\eta_1 t}{\operatorname{sh}\eta_1\Lambda} - \frac{\operatorname{sh}\eta_2 t}{\operatorname{sh}\eta_2\Lambda} \tag{2.13}$$

We denote by $\sigma(\varepsilon, U)$ the set of EVs $P_n^{(j)}$ of problem (2.1) which satisfy the inequality

$$(P_n^{(j)}-1)/\varepsilon \leqslant U,\quad 0<U\sim O(1),\quad j=0,1;\quad n=1,2,\dots \tag{2.14}$$

The set of VEFs corresponding to the EVs in $\sigma(\varepsilon, U)$ are denoted by $H(\varepsilon, U)$. We have the following relations

$$\begin{aligned}\sigma(\varepsilon,U)&=\sigma^{(0)}(\varepsilon,U)\cup\sigma^{(1)}(\varepsilon,U)\\ H(\varepsilon,U)&=H^{(0)}(\varepsilon,U)\cup H^{(1)}(\varepsilon,U)\end{aligned} \tag{2.15}$$

Here the set $H^{(j)}(\varepsilon, U)$ corresponds to the EV set $\sigma^{(j)}(\varepsilon, U)$. The VEFs $S \in H(\varepsilon, U)$ satisfy

$$\mathbf{S}\in\begin{cases}H^{(0)}(\varepsilon,U), & \text{if}\quad S_x\mathbf{S}=iS_y\mathbf{S}=\mathbf{S}\\ H^{(1)}(\varepsilon,U), & \text{if}\quad -S_x\mathbf{S}=iS_y\mathbf{S}=\mathbf{S}\ (i=\pm1)\end{cases} \tag{2.16}$$

$$(S_x:\ \mathbf{u}(x,y)\to\mathbf{u}(-x,y),\quad S_y:\ \mathbf{u}(x,y)\to\mathbf{u}(x,-y))$$

3. APPLICATION OF THE LYAPUNOV–SCHMIDT METHOD

We shall use the operator form of the Lyapunov–Schmidt method [2] to investigate the branching of the trivial solution ($\xi = 0$) of Eq. (1.3) and to construct new solutions in the neighbourhood of any bifurcation point $P_0 \in \sigma(\varepsilon, U)$. Putting $\mathbf{u} = \mathbf{x}$, $P = P_0 + \lambda$, from (1.3) we obtain operator equations for small perturbations $\mathbf{x} = (\omega, \psi)$, λ

$$M(P_0,\varepsilon^2)\mathbf{x}=\Pi\mathbf{x}+\lambda T_1\mathbf{x}+\xi T_2(\zeta)\mathbf{x}+P_0\xi T_1\zeta+\lambda\xi T_1\zeta \tag{3.1}$$

$$\Pi\mathbf{x}=\left\|\begin{matrix}[\omega,\psi]\\ [\omega,\omega]/2\end{matrix}\right\|,\quad T_1()=\left\|\begin{matrix}()_{,xx}\\ 0\end{matrix}\right\|,\quad T_2(\zeta)\mathbf{x}=\left\|\begin{matrix}[\zeta,\psi]\\ [\zeta,\omega]\end{matrix}\right\|$$

It can be shown that the operators $M(P_0, \varepsilon^2)$, Π, T_1, T_2 act from H_4 into Y_2, and moreover that the operator $M(P_0, \varepsilon^2)$ is formally self-adjoint. Because P_0 is a simple critical value (at least for small $\varepsilon > 0$) of the operator $M(P_0, \varepsilon^2)$, we construct [2] a Schmidt operator M_1 in the form

$$M_1\mathbf{x}=M(P_0,\varepsilon^2)\mathbf{x}+\mu\mathbf{S}_0,\quad \mu=\langle\mathbf{x},\kappa\rangle_1 \tag{3.2}$$

Here $\langle\mathbf{x}, \kappa\rangle_1$ is the value of the functional $\kappa \in (H_4)^*$ at the element $\mathbf{x} \in H_4$ and $\mathbf{S}_0$ is the VEF corresponding to the EV P_0. We assume that

$$\langle\mathbf{S}_0,\mathbf{S}_0\rangle_4=\kappa_1;\qquad \langle\mathbf{S}_0,\mathbf{S}_0\rangle_2=\kappa_2 \tag{3.3}$$

We note that taking (3.3) into account the relation

$$\langle\,\mathbf{x},\kappa\,\rangle_1=\langle\,M_1\mathbf{x},\mathbf{S}_0\,\rangle_2/\kappa_2 \tag{3.4}$$

follows from (3.2).

We write (3.2) in the form of the equivalent system of equations

$$\begin{aligned}&M_1\mathbf{x}=\Pi\mathbf{x}+\lambda T_1\mathbf{x}+\xi T_2(\zeta)\mathbf{x}+P_0\xi T_1\zeta+\lambda\xi T_1\zeta+\mu\mathbf{S}_0\\ &\mu=\langle\,\mathbf{x},\kappa\,\rangle_1\end{aligned} \tag{3.5}$$

We shall seek a solution of system (3.5) in the form of a series of integer powers of the small parameters μ, λ, ξ

$$\mathbf{x}=\mathbf{x}_{100}\mu+\mathbf{x}_{010}\lambda+\mathbf{x}_{001}\xi+\sum_{i+j+k\geqslant2}\mathbf{x}_{ijk}\mu^i\lambda^j\xi^k\qquad \mathbf{x}_{ijk}=(\omega_{ijk},\psi_{ijk}) \tag{3.6}$$

Substituting (3.6) into the first equation of (3.5) and equating coefficients of like powers of the parameters μ, λ, ξ, we obtain linear operator equations for determining $\mathbf{x}_{ijk}$ which are solvable by virtue of the generalized Schmidt lemma [2]

$$M_1\mathbf{x}_{ijk} = \mathbf{f}_{ijk}, \quad i+j+k \geqslant 1; \quad \mathbf{x}_{ijk} \in H_4, \quad \mathbf{f}_{ijk} \in Y_2 \tag{3.7}$$

Here, in particular

$$\mathbf{f}_{100} = \mathbf{S}_0, \quad \mathbf{f}_{010} = 0, \quad \mathbf{f}_{001} = P_0 T_1 \zeta$$
$$\mathbf{f}_{200} = \Pi \mathbf{x}_{100}, \quad \mathbf{f}_{300} = \left\| \begin{matrix} [\omega_{100}, \psi_{200}] + [\omega_{200}, \psi_{100}] \\ [\omega_{100}, \omega_{200}] \end{matrix} \right\|$$

We will find the first few coefficients of series (3.6). Using $M_1\mathbf{S}_0 = \kappa_1\mathbf{S}_0$ we obtain $\mathbf{S}_0 = \Gamma\mathbf{S}_0$ where $\Gamma = (M_1)^{-1}$. From this we find that $\mathbf{x}_{100} = \mathbf{S}_0/\kappa_1$. Then from condition $\mathbf{f}_{010} = 0$ we have $\mathbf{x}_{010} = 0$. Sequentially solving Eqs (3.7) we find

$$\mathbf{x}_{001} = P_0\Gamma(T_1\zeta), \quad \mathbf{x}_{110} = \Gamma(T_1\mathbf{x}_{100}) = \Gamma(T_1\mathbf{S}_0)/\kappa_1 \tag{3.8}$$
$$\mathbf{x}_{200} = \Gamma(\Pi\mathbf{x}_{100}) = \Gamma(\Pi\mathbf{S}_0)/\kappa_1^2$$

etc.

Noting that

$$\langle \mathbf{x}_{ijk}, \kappa \rangle_1 = \langle M_1\mathbf{x}_{ijk}, \mathbf{S}_0 \rangle_2 / \kappa_2 = \langle \mathbf{f}_{ijk}, \mathbf{S}_0 \rangle_2 / \kappa_2 \tag{3.9}$$

we obtain a numerically more convenient form of the equations governing $\mathbf{x}_{001}$, $\mathbf{x}_{200}$

$$\begin{aligned} M(P_0, \varepsilon^2)\mathbf{x}_{001} &= P_0(T_1\zeta - \langle T_1\zeta, \mathbf{S}_0\rangle_2 \mathbf{S}_0/\kappa_2) \\ M(P_0, \varepsilon^2)\mathbf{x}_{200} &= (\Pi\mathbf{S}_0 - \langle \Pi\mathbf{S}_0, \mathbf{S}_0\rangle_2 \mathbf{S}_0/\kappa_2)/\kappa_1^2 \end{aligned} \tag{3.10}$$

Substituting series (3.6) with known coefficients into the second relation of (3.5), we derive the branching equation, which can be written in the form

$$\begin{aligned} &L_{300}\mu^3 + L_{110}\lambda\mu + L_{100}\xi + \ldots = 0 \\ &L_{001} = P_0\langle \zeta_{,x}, \omega_{0,x}\rangle, \quad L_{110} = \langle \omega_{0,x}, \omega_{0,x}\rangle/\kappa_1 \\ &L_{300} = (\langle [\omega_0, \psi_{200}] + [\omega_{200}, \psi_0], \omega_0\rangle + \langle [\omega_0, \psi_{200}], \psi_0\rangle)/\kappa_1 \end{aligned} \tag{3.11}$$

The dots denote terms that are of higher order in μ, λ, ξ, and the angle brackets denote the scalar product of functions.

We note that $L_{200} = 0$ by virtue of the periodicity of the VEF $\mathbf{S}_0$ in the variable y. We assume that $L_{300} \neq 0$ and $L_{001} = 0$. Then (3.11) can be transformed into the form

$$\begin{aligned} &\Phi(\mu, \lambda, \xi) \equiv b\mu^3 - \lambda\mu + K(\zeta)\xi + \ldots = 0 \\ &b = -L_{300}/L_{110}, \quad K(\zeta) = -L_{001}/L_{110} \end{aligned} \tag{3.12}$$

According to Koiter the coefficient b (the Koiter parameter) is a coefficient of the sensitivity of the structure to imperfections. That is, if $b < 0$, the shell is taken to be sensitive to imperfections, because in this case the critical stability-loss load for an imperfect shell $P_{0,s}$ is less than that for the perfect shell P_0. The limit point $P_{0,s}$ is found from the simultaneous solution of Eqs (3.12) and

$$\partial\Phi(\mu, \lambda_s, \xi)/\partial\mu = 0, \quad \lambda_s = P_{0,s} - P_0 \tag{3.13}$$

and is obtained in the form

$$P_{0,s} = P_0 + 3(K(\zeta)\xi/2)^{2/3} b^{1/3} + \ldots \tag{3.14}$$

We remark that formula (3.14) was first obtained by Koiter. The solution of Eq. (1.3) corresponding to the limiting point $P_{0,s}$ is unstable. Because of what has been said earlier the problem of establishing the sign of the Koiter parameter b is very important. Few analytic results have been obtained. We note the papers [3, 4] which proved the positivity of the Koiter parameter for plates of arbitrary shape under the action of external pressure. A graph of the dependence of the Koiter parameter on the Batdorf parameter has been given [5] and an asymptotic formula for the parameter b when $Z \to \infty$ has been numerically obtained for a cylindrical shell undergoing axial compression with clamped ends. Below we construct the leading asymptotic term ($Z \to \infty$) of the Koiter parameter corresponding to a cylindrically compressed shell with moveable hinged ends.

4. ASYMPTOTIC FORMS OF THE PARAMETER b WHEN $P_0 \in \sigma(0)(\varepsilon, U)$

We introduce the notation

$$l_{110} = \kappa_1 L_{110}, \quad l_{300} = \kappa_1^3 L_{300} \tag{4.1}$$

We consider the case when $P_0 \in \sigma^{(0)}(\varepsilon, U)$, i.e. $\mathbf{S}_0(t, y) = (u_n^{(0)}, v_n^{(0)})(t)\sin(d_n y + \pi n/2)$ $(n = 1, 2, \ldots)$. With the help of the replacement $x = \varepsilon t$ we deduce from (3.11) that

$$l_{110} = \frac{l}{2\varepsilon} I[(u_n')^2], \quad I[f] = \int_{-\Lambda}^{\Lambda} f(t)dt \tag{4.2}$$

Here we omit the superscript (0) from u_n. Using the asymptotic formulae

$$\eta_1 = \frac{1}{2}(1 + i\sqrt{3}) + O(\varepsilon^2), \quad \eta_2 = O(\varepsilon^2), \quad \eta = \frac{1}{2}(1 + i\sqrt{3}) + O(\varepsilon^2)$$

$$C = -\frac{1}{2}(1 - i\sqrt{3}) + O(\varepsilon^2), \quad Q = -\frac{1}{2}(1 + i\sqrt{3}) + O(\varepsilon^2), \quad J = \left(1 + \frac{i}{\sqrt{3}}\right) + O(\varepsilon)$$

we find from (4.2) that

$$l_{110} = 4l/\varepsilon + O(1) \quad (n = 1, 2, \ldots) \tag{4.3}$$

Thus the leading term in the asymptotic expansion of l_{110} does not depend on the wave number n along the circumference of the cylindrical shell.

We now construct an asymptotic formula for κ_1. Noting that the relation

$$P_n = \|\mathbf{S}_n\|_4^2 / (\varepsilon l_{110})$$

follows from (2.1) and taking the limit as $\varepsilon \to 0$ in this relation using (4.3), we find

$$\kappa_1 = 4l + O(\varepsilon) \tag{4.4}$$

This means that

$$L_{110} = 1/\varepsilon + O(1) \tag{4.5}$$

To construct the asymptotic expansion of L_{300} it is first necessary to construct the solution of the boundary-value problem

$$\begin{aligned}
&\Delta_1^2\omega_{200} + P_n\omega_{200}'' - \psi_{200}'' = -d_n^2[(u_n\upsilon_n)'' + (-1)^{n+1}\Phi_1\cos 2d_n y]/2 \\
&\Delta_1^2\psi_{200} + \omega_{200}'' = d_n^2[(u_n^2/2)'' + (-1)^{n+1}\Phi_2\cos 2d_n y]/2
\end{aligned} \tag{4.6}$$

$$\Delta_1(\ ) \equiv (\ )'' + (\ )_{,yy} \quad (\ )' = \partial(\ )/\partial t$$

$$\omega_{200}(\pm\Lambda) = 0, \quad \omega_{200}''(\pm\Lambda) = 0, \quad \psi_{200}(\pm\Lambda) = 0, \quad \psi'(\pm\Lambda) = 0$$

$$\Phi_1 = u_n''\upsilon_n - 2u_n'\upsilon_n' + u_n\upsilon_n'', \quad \Phi_2 = u_n''u_n - (u_n')^2$$

which is obtained from the second equation of (3.10) with

$$S_0(t, y) = (u_n, \upsilon_n)(t)\sin(d_n y + \pi n/2)$$

without using κ_1. In formulae (4.6) the superscript (0) has been omitted from P_n, u_n, υ_n, ω_{200}, ψ_{200}. We shall seek the solution of problem (4.6) in the form

$$\begin{aligned}
&\omega_{200}(t,y) = \omega_1(t) + (-1)^n\omega_2(t)\cos 2d_n y \\
&\psi_{200}(t,y) = \psi_1(t) + (-1)^n\psi_2(t)\cos 2d_n y
\end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6), and separating the variables, we obtain the boundary-value problem

$$\begin{aligned}
&\Phi(\alpha_i)\omega_i + P_n\omega_i'' - \psi_i'' = f_i, \quad \Phi(\alpha_i)\psi_i + \omega_i'' = g_i \\
&\omega_i(\pm\Lambda) = 0, \quad \omega_i'(\pm\Lambda) = 0, \quad \psi_i(\pm\Lambda) = 0, \quad \psi_i'(\pm\Lambda) = 0
\end{aligned} \tag{4.8}$$

$$\Phi(\alpha)(\ ) \equiv (\ )^{\mathrm{IV}} - 2\alpha^2(\ )'' + \alpha^4(\ ), \quad \alpha_1 = 0, \quad \alpha_2 = 2\theta_n$$

$$f_1 = -\frac{d_n^2}{2}(u_n\upsilon_n)'', \quad f_2 = \frac{d_n^2}{2}\Phi_1, \quad g_1 = \frac{d_n^2}{2}\left(\frac{u_n^2}{2}\right)'', \quad g_2 = -\frac{d_n^2}{2}\Phi_2$$

governing $\omega_i(t)$, $\psi_i(t)$ $(i = 1, 2)$.

Using relations (3.11) and (4.7), we transform the formula for l_{300} into the form

$$l_{300} = -\frac{d_n^2 l}{2\varepsilon}I[2\omega_1'u_n\upsilon_n + \psi_1'u_n^2] + \frac{l}{\varepsilon}I[f_2\omega_2 + g_2\psi_2] \tag{4.9}$$

We construct the asymptotic solutions of boundary-value problems (4.8) using the asymptotic integration method [6]. We note that according to the results of Section 2 we have the asymptotic representations

$$P_n^{(0)} = \sum_{i=0}^{\infty} P_{n,i}^{(0)}\varepsilon^i, \quad P_{n,0}^{(0)} = P_0^{(0)}; \quad P_{n,1}^{(0)} = 0, \quad P_{n,2}^{(0)} = d_n^2 \quad (n = 1, 2, \ldots) \tag{4.10}$$

The right-hand sides of boundary-value problems (4.8) can also be represented in the form of series

$$f_i(t) = \sum_{j=0}^{\infty} f_i^{(j)}(t)\varepsilon^j, \quad g_i(t) = \sum_{j=0}^{\infty} g_i^{(j)}(t)\varepsilon^j$$

where $f_i^{(0)}(t) \not\equiv 0$, $g_i^{(0)}(t) \not\equiv 0$. We seek the solutions of boundary-value problems (4.8) in the form of series

$$\begin{aligned}
&\omega_i(t) = \varepsilon^{-2}\omega_i^{(-2)}(t) + \varepsilon^{-1}\omega_i^{(-1)}(t) + \omega_i^{(0)}(t) + \ldots \\
&\psi_i(t) = \varepsilon^{-2}\psi_i^{(-2)}(t) + \varepsilon^{-1}\psi_i^{(-1)}(t) + \psi_i^{(0)}(t) + \ldots \quad (i = 1, 2)
\end{aligned}$$

Substituting these series into (4.8) and equating the coefficients of ε^j to zero, we obtain boundary-value problems governing $\omega_i^{(j)}(t)$, $\psi_i^{(j)}(t)$. We solve these for $j = -2$ and obtain

$$\omega_i^{(-2)}(t) = C_i^{(-2)}u_0^{(0)}(t), \quad \psi_i^{(-2)}(t) = C_i^{(-2)}\upsilon_0^{(0)}(t) \quad (i = 1,2) \tag{4.11}$$

The constants $C_i^{(-2)}$ are found from the solvability conditions for these boundary-value problems when $j = 0$, and are obtained in the form

$$\begin{aligned} C_1^{(-2)} &= -\frac{1}{2}\frac{I[2u_0u_0'\upsilon_0' - (u_0')^2\upsilon_0]}{I[(\upsilon_0')^2]} + O(\varepsilon) \\ C_2^{(-2)} &= -\frac{3}{2}\frac{I[2u_0u_0'\upsilon_0' - (u_0')^2\upsilon_0]}{I[7(u_0')^2 - 8(\upsilon_0')^2]} + O(\varepsilon) \end{aligned} \tag{4.12}$$

(the superscript (0) is omitted from u_0, υ_0). Using relation (4.3) in (2.12), from (4.12) we deduce that

$$C_i^{(-2)} = -1 + O(\varepsilon)$$

From this we obtain

$$\omega_i(t) = -u_0^{(0)}(t)\varepsilon^{-2} + O(\varepsilon^{-1}), \quad \psi_i(t) = -\upsilon_0^{(0)}(t)\varepsilon^{-2} + O(\varepsilon^{-1}) \quad (i = 1,2)$$

We now use the latter formulae together with relation (4.9) and compute the integrals, obtaining

$$L_{300} = \frac{d_n^2}{8l^2}\varepsilon^{-3} + O(\varepsilon^{-2}), \quad b = -\frac{d_n^2}{8l^2}\varepsilon^{-2} + O(\varepsilon^{-1}) \tag{4.13}$$

From this it follows that for sufficiently small $\varepsilon > 0$ the bifurcation point $P_n^{(0)} \in \sigma^{(0)}(\varepsilon, U)$ turns into the limit point $P_{n,s}^{(0)}$ where

$$P_{n,s}^{(0)} = P_n^{(0)} - \frac{3}{2}\left(\frac{K(\zeta)d_n\xi}{l\varepsilon}\right)^{2/3} + \ldots \tag{4.14}$$

Here the dots denote terms of higher order of smallness in ξ; it is assumed that $\xi = O(\varepsilon^2)$, i.e. the amplitude of the initial imperfections is proportional to the relative thinness of the shell.

We introduce the notation

$$P_s = \min_n P_{n,s}^{(0)} \tag{4.15}$$

and call P_s the limit point corresponding to the set of bifurcation points $P_n^{(0)} \in \sigma^{(0)}(\varepsilon, U)$.

With an appropriate choice of the function ζ the limit point P_s can correspond to any bifurcation point from the set $\sigma^{(0)}(\varepsilon, U)$.

Indeed, suppose

$$\zeta(x,y) = \sum_i \alpha_i W_i(x,y)$$

where $W_i(x, y)$ are the first coordinates of the VEF of boundary-value problem (2.1) corresponding to the EV P_i. Using the orthogonality condition

$$\langle W_i, W_{j,xx} \rangle = 0, \quad i \neq j$$

we derive $L_{001} = P_n^{(0)}\alpha_n l_{110}$ from (3.11).

From this we have

$$K(\zeta) = -P_n^{(0)}\alpha_n / \kappa_1 = -\alpha_n / (4l) + O(1)$$

Then

$$P_{n,s}^{(0)} = 1 - \frac{3}{2l}\left(\frac{\pi n \alpha_n \xi}{2\varepsilon}\right)^{2/3} + \dots \qquad (4.16)$$

Now suppose that $\zeta(x, y) = 2W_j(x, y)/(\pi j)$. Then the relation

$$P_s = P_{j,s}^{(0)} = 1 - \frac{3}{2l}\left(\frac{\xi}{\varepsilon}\right)^{2/3} + \dots \qquad (4.17)$$

follows from (4.15) and (4.16).

The result obtained above can be expressed as a theorem.

Theorem 2. Let $P_n^{(0)} \in \sigma^{(0)}(\varepsilon, U)$ $\xi \sim O(\varepsilon^2)$ $(\varepsilon \to 0)$. Then to the bifurcation point P_n there corresponds a limit point $P_{n,s} < P_n$ with asymptotic representation (4.14). Moreover, for every j such that $P_j \in \sigma^{(0)}(\varepsilon, U)$, one can select an imperfection function ζ such that $P_{s,j}$ is the limit point corresponding to the set of bifurcation points $\sigma^{(0)}(\varepsilon, U)$.

5. ASYMPTOTIC FORMS OF THE PARAMETER b WHEN $P_0 \in \sigma^{(1)}(\varepsilon, U)$

In this case $S_0(t, y) = (u_n^{(1)}, v_n^{(1)})(t) \sin(d_n y + \pi n/2)$ $(n = 1, 2, \dots)$. Using the asymptotic formulae

$$\eta_1 = \frac{1}{2}(1 + i\sqrt{3}) + \frac{1}{2}(1 + i\sqrt{3})\varepsilon + O(\varepsilon^2), \quad \eta_2 = O(\varepsilon^2),$$

$$C = -\frac{1}{2}(1 - i\sqrt{3}) + (1 - i\sqrt{3})\varepsilon + O(\varepsilon^2)$$

$$Q = \frac{1}{2}(1 + i\sqrt{3}) + 2\left(1 - \frac{i}{\sqrt{3}}\right)\varepsilon + O(\varepsilon^2), \quad J = \left(1 - \frac{i}{\sqrt{3}}\right) - (1 - i\sqrt{3})\varepsilon + O(\varepsilon^2)$$

and continuing as in Section 4, we again arrive at formulae (4.4) and (4.5) for κ_1, l_{110}.

In the same way as above we obtain

$$L_{300} = -\frac{5d_n^4}{8l^2}\varepsilon^{-2} + O(\varepsilon^{-1}), \quad b = \frac{5d_n^4}{8l^2}\varepsilon^{-1} + O(1) \qquad (5.1)$$

It follows from (5.1) that $b > 0$. Consequently, the bifurcation points $P_n^{(1)}$ do not turn into limit points, and the corresponding eigenshapes (odd with respect to the axial coordinate) are stable.

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